Cosmological stretching of perturbations on a cosmic string

Christian Stephan-Otto † Ken D. Olum † and Xavier Siemens ‡

† Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, MA 02155, USA

‡ Center for Gravitation and Cosmology, Department of Physics, University of Wisconsin — Milwaukee, P.O. Box 413, Wisconsin 53201, USA

E-mail: `christian@cosmos.phy.tufts.edu`, `kdo@cosmos.phy.tufts.edu`, `siemens@gravity.phys.uwm.edu`

Abstract. We investigate the effects of cosmological expansion on the spectrum of small-scale structure on a cosmic string. We simulate the evolution of a string with two modes that differ in wavelength by one order of magnitude. Once the short mode is inside the horizon, we find that its physical amplitude remains unchanged, in spite of the fact that its comoving wavelength decreases as the longer mode enters the horizon. Thus the ratio of amplitude to wavelength for the short mode becomes larger than it would be in the absence of the long mode.

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1. Introduction

Particle physics models that involve symmetry breaking phase transitions [1, 2] as well as some brane-world scenarios [3] predict the formation of topological defects. Much work has gone into understanding the evolution of defects (see, for example, [2]). Among defects, cosmic strings have drawn the bulk of the effort because, unlike monopoles and domain walls, they do not cause cosmological disasters. Indeed, strings are viable candidates for a variety of interesting cosmological phenomena such as gamma ray bursts [4], gravitational wave bursts [5] and ultra high energy cosmic rays [6]. The detailed predictions for these phenomena, however, depend sensitively on the spectrum of perturbations present on cosmic strings.

The result of a phase transition that produces strings is a network of long strings that stretch across the horizon and a collection of closed loops. The large scale evolution of a string network is well understood analytically [7, 8, 9, 10] as well as numerically [11, 12, 13]. Cosmic string networks quickly evolve toward a "scaling" regime in which correlation lengths of long strings and average sizes of loops scale with the cosmic time t. This solution is made possible by intersections of long strings which produce loops. Gravitational radiation causes loops to decay and so reduces the total length of string in the network. There is much structure present on the long strings arising from the Brownian character of the network at formation as well as a build-up of the sharp edges (kinks) [14] produced at string intersections.

The lower bound on the size of the structure, also referred to as the small-scale structure cutoff, is thought to be given by gravitational back-reaction. It turns out that a given mode of a perturbation on a string interacts gravitationally only with a narrow range of other modes [15, 16]. The range depends on the amplitudes and wavelengths, and hence the spectrum, of the modes that make up the perturbations. The result of this is that the small-scale structure cutoff depends rather sensitively on the spectrum of perturbations present on strings.

The spectrum of perturbations, in turn, depends on a combination of two effects: 1) the cosmological stretching of the primordial perturbations present on the strings, and 2) the formation and build-up of kinks due to string intersections.

In [16] a simple model to account for cosmological stretching was used. In this model all modes with wavelengths smaller than the horizon are stretched: their wavelengths grow with the scale factor, while their physical amplitudes are unchanged. In this work we show that this is only true of modes with wavelengths smaller than, but comparable to, the horizon. The evolution of much smaller modes is more complicated: their physical amplitudes again remain unchanged, but while they are being stretched by the expansion of the Universe a change in the arc length of the string acts to decrease their comoving wavelength.

In Section II we review the motion of strings in expanding space-time. In Section III we describe the computational scheme as well as initial conditions used in our simulations and we discuss the results. We conclude in Section IV.

2. Motion of strings in expanding space-time

If the typical length scale of a cosmic string is large compared to its thickness, the string can be accurately modeled by a one-dimensional object. The equations of motion are obtained from the Nambu-Goto action, which is proportional to the area swept by the world-sheet of the string.

We consider the dynamics of strings in an expanding FRW space-time with a line element $ds^2 = a^2(\tau)(d\tau^2 - d\mathbf{x}^2)$. The comoving spatial coordinates of the string, $\mathbf{x}(\tau,\sigma)$, are written as functions of the conformal time τ and a spatial parameter σ.

It is convenient to choose a gauge in which the unphysical parallel components of the velocity vanish

$$\dot{\mathbf{x}} \cdot \mathbf{x}' = 0. \tag{1}$$

In this gauge the equations of motion of the string are [17]

$$\ddot{\mathbf{x}} + 2\left(\frac{\dot{a}}{a}\right)\dot{\mathbf{x}}\left(1 - \dot{\mathbf{x}}^2\right) = \left(\frac{1}{\epsilon}\right)\left(\frac{\mathbf{x}'}{\epsilon}\right)' \tag{2}$$

with

$$\epsilon = \sqrt{\frac{\mathbf{x}'^2}{1 - \dot{\mathbf{x}}^2}} \tag{3}$$

and

$$\frac{\dot{\epsilon}}{\epsilon} = -2\,\frac{\dot{a}}{a}\,\dot{\mathbf{x}}^2. \tag{4}$$

Dots and primes denote derivatives with respect to τ and σ respectively.

The string's total energy is given by $a(\tau)\mu\int \epsilon d\sigma$, where μ is the string's mass per unit length, so that ϵ can be thought of as the energy per unit σ in comoving units. There remains some freedom in the choice of parameterization which we can use to set $\epsilon = 1$ in the initial conditions.

In flat space-time Equation (2) reduces to the wave equation. The general solution can be written as the superposition of oppositely moving waves

$$\mathbf{x}(\tau,\sigma) = \frac{1}{2}[\mathbf{a}(\sigma-\tau) + \mathbf{b}(\sigma+\tau)] \tag{5}$$

with the constraints

$$\mathbf{a}'^2 = \mathbf{b}'^2 = 1 \tag{6}$$

We can express the functions $\mathbf{a}'$ and $\mathbf{b}'$ in terms of $\dot{\mathbf{x}}$ and $\mathbf{x}'$

$$\mathbf{a}'(\tau,\sigma) = \mathbf{x}'(\tau,\sigma) - \dot{\mathbf{x}}(\tau,\sigma) \tag{7}$$

and

$$\mathbf{b}'(\tau,\sigma) = \mathbf{x}'(\tau,\sigma) + \dot{\mathbf{x}}(\tau,\sigma). \tag{8}$$

Although these equations are derived for the flat space-time case, they turn out to be useful for the reconstruction of the string shape in curved space-time.

As in [18, 2], we consider perturbations on a straight static string

$$\mathbf{x}(\tau,\sigma) = \mathbf{c}\sigma + \delta\mathbf{x}(\tau,\sigma) \tag{9}$$

with $\mathbf{c}$ constant. In an FRW space-time with power-law expansion $a(\tau) = \tau^\alpha$ the solution to Equations (2) and (4) for small perturbations is a superposition of waves of the form [18, 2]

$$\delta\mathbf{x}(\tau,\sigma) = \mathbf{A}\tau^{-\nu}J_\nu(\kappa\tau)e^{i\kappa\sigma}. \tag{10}$$

where $\mathbf{A}\cdot\mathbf{c} = 0$ and $\nu = \alpha - 1/2$. The physical wavelength of the perturbations is given by

$$\lambda = a(\tau)\frac{2\pi}{\kappa} \tag{11}$$

and therefore $\kappa\tau = 2\pi\tau^{\alpha+1}/\lambda \sim t/\lambda$, is approximately the ratio of the horizon to the size of the mode. The behavior that follows from Equation (10) is rather simple.

When the wavelength of the mode is large compared to the horizon, $\kappa\tau \ll 1$, the small argument expansion for the Bessel function can be used to show that the comoving amplitude is constant in time, namely,

$$\delta\mathbf{x} \approx \mathbf{A}\left(\frac{\kappa}{2}\right)^\nu \frac{e^{i\kappa\sigma}}{\Gamma(\nu+1)}. \tag{12}$$

Therefore both the physical wavelength and amplitude are proportional to the scale factor and the overall size of the string grows while its shape remains fixed.

When the mode is well inside the horizon, $\kappa\tau \gg 1$, the large argument expansion of the Bessel function can be used to show that the comoving amplitude goes as the inverse of the scale factor

$$\delta\mathbf{x} \approx \mathbf{A}\tau^{-\alpha}\sqrt{\frac{2}{\pi\kappa}}\cos(\kappa\tau - \alpha\pi/2)e^{i\kappa\sigma}. \tag{13}$$

Therefore the physical amplitude is constant while the wavelength grows. This results in a straightening of the mode: the amplitude to wavelength ratio decreases with the scale factor.

If we ignore any interaction between different perturbations, it is easy to show [16] that we obtain a power-law spectrum for the amplitude to wavelength ratios. Shorter perturbations have been inside the horizon longer, and therefore have been proportionately more damped than longer ones.

However, if there are simultaneous perturbations of different wavelengths, and their amplitudes are not too small, we expect interactions between them. The effect on the wavelengths of the perturbations can be understood without simulation. Let us consider a string with two perturbations, one with wavelength much less than the other. Suppose that the long excitation is in the form of a standing wave, so that there are periodically times when it does not contribute to the string motion. At such time, we can easily measure the wavelength of the short excitation which is just the length of one cycle along the underlying string.

The number of cycles of the short excitation in a comoving box is unchanged by the expansion of the universe, but the arc length of the underlying string is decreased because the amplitude of the long excitation is damped. Thus the comoving wavelength of the short excitations decreases, since the same number of them fit on a string with shorter comoving arc length. In terms of physical length, we expect the short wavelength to still increase, but not as much as it would have, had the long excitation not been present.

On the other hand, the evolution of the amplitude of the short excitation is not at all obvious. One might think that when the horizon is much larger than the short wavelength, the short excitation physical amplitude remains unchanged, or one might think that the amplitude to wavelength ratio of the short excitations evolves in the same way (going inversely with the scale factor) as if the long excitations were not present. To determine which of these ideas is correct, we turn to numerical simulations.

3. Numerical simulations of strings in expanding space-time

In our simulations the strings evolve according to discretized versions of Equations (2) and (4). Because of its simplicity we have chosen the high-resolution leapfrog scheme described in [12].

In order to avoid the numerical instabilities of the scheme, the initial conditions must be such that there are no discontinuities (no kinks) and the velocity of the string is never close to the speed of light (no cusps). For our purposes it is sufficient to investigate the evolution of an initially static cosmic string, with fixed ends and two modes present. We will use transverse modes to simplify the retrieval of information about their amplitudes.

To construct the initial string we start with a single cycle of a sinusoid ‡ $y = A\sin(2\pi z/L)$, let s be its arc length parameter such that $s \in [0, S]$ and let $x = \alpha\sin(2\pi m s/S)$. The string's shape is defined then by

$$\mathbf{x}(\tau = 0, z) = (\alpha\sin\frac{2\pi m s}{S}, A\sin\frac{2\pi z}{L}, z). \tag{14}$$

This can be conceptualized as m cycles of a short mode of amplitude α and wavelength $\lambda = S/m$ that rest on a paper band, which is then deformed to match its long edge with a simple sinusoid of amplitude A and wavelength L (see Figure 1). Once the values for the string parameters have been chosen and the string has been constructed we reparametrize it with respect to the usual arc length $\sigma \in [0, \Sigma]$.

To compare the evolution of the amplitude of the short mode in the previous case to one in which the long mode is not present we evolve a second set of initial conditions. In this case the static string is given by

$$\mathbf{x}(\tau = 0, z) = (\alpha\sin\frac{2\pi m z}{S}, 0, z) \tag{15}$$

‡ Note that y is a sinusoid in z, not in string length σ. If one uses σ instead of z one finds a maximum amplitude $A = 1$ and with that value parts of the string (in flat spacetime) move at the speed of light.

where $z \in [0, S]$. These initial conditions are such that the length of the string along the z-direction S is the arc-length of the long mode above. In terms of the paper band analogy, in this case the band has been flattened out to lie entirely on the x-z plane (bottom of Figure 1).

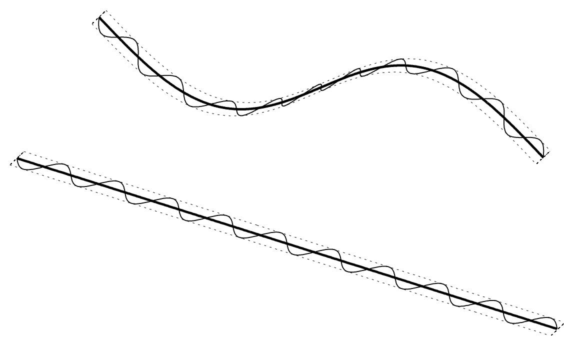

Figure 1. Initial shape of $\mathbf{x}$ according to the "paper band" construction. The thick solid line depicts the long mode $y = A \sin 2\pi z/L$ in the two-mode case and a straight line in the single mode case.

We have performed a number of simulations to investigate the parameter space of the initial conditions for the string. Several runs where carried out for radiation- as well as matter-dominated universes. These lead to no substantial differences in the behavior. The figures we show below, which correspond to a universe dominated by radiation, are representative of all the studied cases.

As in [12], we check the accuracy of our results by numerically evolving ϵ through the discretized version of Equation (4) and calculating it independently from the numerical results for $\dot{\mathbf{x}}$ and $\mathbf{x}'$ and Equation (3). In our simulations the evolved ϵ is in agreement with the value computed from $\mathbf{x}'$ and $\dot{\mathbf{x}}$ to within 0.1% for all cases. The initial number of spatial oscillations along the string is conserved, meaning that the equations of motion do not lead to the generation of new modes.

We set the initial ratios of amplitude to angular wavelength, $\mathcal{E}_l = 2\pi A/L$ and $\mathcal{E}_s = 2\pi\alpha/\lambda$, for the long and short modes respectively, to be unity. These quantities characterize the maximum slope of each mode on the string. We take $m = 10$ cycles of the short mode, and choose the initial amplitude of the long mode to be $A = 100$ in arbitrary units. The wavelength of the long mode then becomes $L = 200\pi$ and the elliptic integral for the arc length yields $S \approx 764.04$, which gives $\lambda \approx 76.40$ and $\alpha \approx 12.16$.

For convenience we chose to simulate the string by a series of points with the comoving spatial distance between each and the next exactly 1 at the initial time. To make that possible we decrease α and thus $\mathcal{E}_s$ by 0.7%, obtaining 928 as the total number of points.

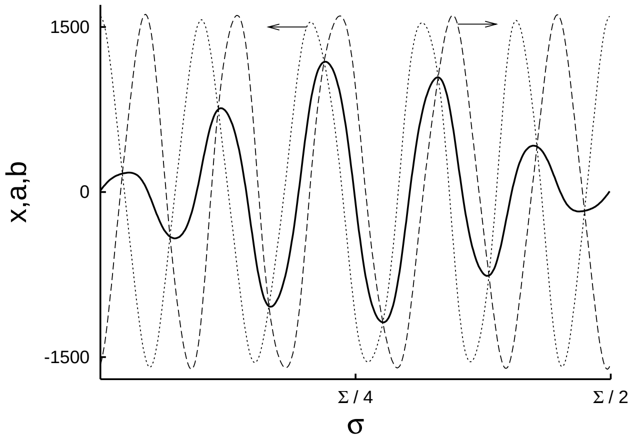

Figure 2. Physical x-direction perturbations of $\mathbf{x}$ and its traveling components $\mathbf{a}$ and $\mathbf{b}$ (solid, dashed and dotted lines respectively) as functions of σ, in a radiation-dominated universe. Only the first half of the string is shown as the second is its repetition. The arrows tell us that when the snapshot was taken neighboring peaks in $\mathbf{a}$ and $\mathbf{b}$ were receding from each other, indicating a decrease in the physical $\mathbf{x}$ perturbation. The snapshot was taken at $\tau = 194.6$.

In order to understand the behavior of the amplitude of the short mode as a function of time we require a means of extracting its amplitude. As the simulation progresses, the shape of the short mode can be seen to vary along the string, as shown in Figure 2, so that it is not obvious what its amplitude is. The reason for this variation is that the maxima and minima of the oscillations of the short mode are reached at different times depending on whether they were located at nodes or anti-nodes of the long mode initially.

To determine the amplitude we compute $\mathbf{a}$ and $\mathbf{b}$ by numerically integrating Equations (7) and (8). As shown in Figure 2, $\mathbf{a}$ and $\mathbf{b}$ approximately preserve the original comoving shape of the string, except for slight shifts in the distance separating their nodes, and we can readily determine the amplitude of the short mode from either of them. Here the short mode peaks of $\mathbf{a}$ and $\mathbf{b}$ are closer to each other for $\sigma \sim \Sigma/4$ than for σ either small or $\sim \Sigma/2$. Nevertheless, their physical amplitude is constant along the string.

In Figure 3 we show the physical amplitude of the short mode as a function of τ for two cases. For one of them the short mode is the only perturbation on the string, while for the other both the short and long modes are present. The starting time of the simulation is $\tau_i = 0.1$ to ensure it is much smaller than both of the scales defining the string. The corresponding initial value of the scale factor is $a_i = 1$. The general behavior in the two cases is the same and is consistent with the solutions presented in Section II as well the analysis presented in [17]. Thus the amplitude of the short mode is insensitive to the presence of a long one.

The evolution of the physical amplitude of the short mode occurs in three stages. At first, when the wavelength is longer than the horizon, conformal stretching prevails

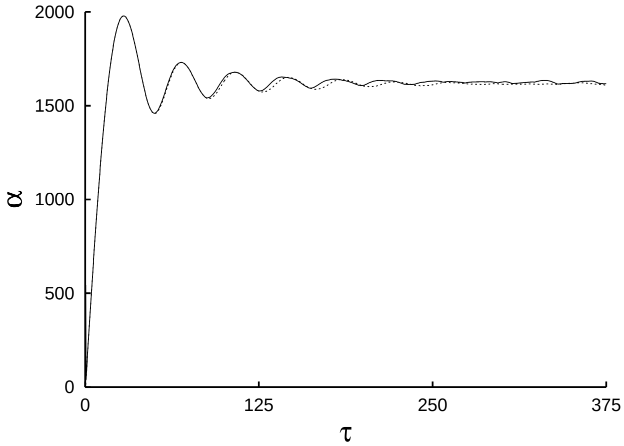

Figure 3. Physical amplitude of the short mode α as a function of τ for a string with both long and short modes (solid line) and for a string with only the short mode (dotted line), in a radiation-dominated universe.

and the amplitude grows with the scale factor, as we expect from Equation (12). The second stage occurs when the short is less than but still comparable to the horizon size. In this stage the amplitude exhibits damped oscillatory behavior. In the final stage the amplitude becomes constant, as expected from Equation (13).

Although the physical amplitude of the short mode is not affected by the presence of the long mode, we have argued in Section II that its wavelength should change as a result of the decreasing arc length of the string. The evolution of the amplitude to wavelength ratio of short modes is therefore sensitive to the presence of long modes.

Figure 4 shows the evolution of $\mathcal{E}_s = 2\pi\alpha/\lambda$ for the cases shown in Figure 3. The wavelength of the short mode evolves differently for the two situations presented. For the string with two modes the comoving arc length of the string decreases with time, causing $\mathcal{E}_s$ to decrease at a somewhat slower rate than in the the single-excitation case.

To make the discrepancy more evident we show in Figure 5 the amplitude to wavelength ratio $\mathcal{E}_s$ multiplied by the scale factor a. For the single-mode case the evolution is exactly the three stage process for the amplitude α described above, because the wavelength of the short mode λ simply scales with the scale factor. On the other hand, the two-mode case shows a more complicated evolution process.

At first both modes are larger than the horizon and conformal stretching prevails; this stage ends with the fall of the short mode into the horizon at $\tau \sim 10$. In the second stage the short perturbation is inside the horizon but the long mode is still stretching conformally, thus the situation is the same as in the single mode case. Once the long mode falls into the horizon its amplitude is damped, causing the arc length of the string to grow at a rate smaller than the scale factor. This is observed at $\tau \sim 50$ where $a\mathcal{E}_s$ begins growing, deviating from the single perturbation case. The final stage is characterized by the amplitude of the long mode being negligible compared to its

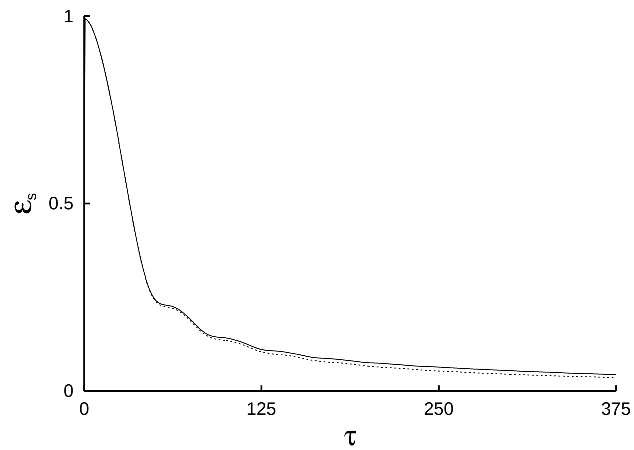

Figure 4. Plot of $\mathcal{E}_s$ for a string with long and short modes (solid line) and for a string with only the short mode (dotted line).

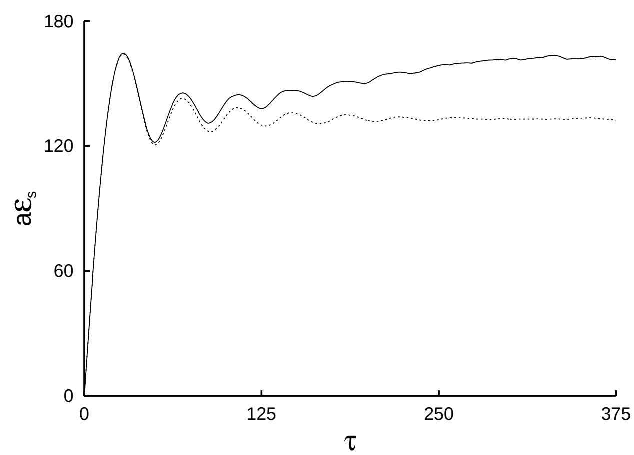

Figure 5. Plot of $a(\tau)\mathcal{E}_s$ for a string with long and short modes (solid line) and for a string with only the short mode (dotted line).

wavelength. Its arc length is then comparable to its wavelength, which scales with a. From then on $a\mathcal{E}_s$ remains constant, albeit larger than in the single perturbation case.

Although here we have only considered the two mode case, in a cosmological setting long modes are always entering the horizon. This means that the fourth stage described above is never reached by a realistic network of strings and the scaled amplitude to wavelength ratio (as in Figure 5) does not stop increasing.

4. Discussion and Conclusions

Astrophysical and cosmological effects of a cosmic string network depend sensitively on the small-scale structure present on the strings. That small-scale structure exists at the time of string formation and is further enhanced by intercommutations. The structure then evolves by cosmological expansion, and gravitational back-reaction smooths out all

structures smaller than a certain cutoff. Gravitational back-reaction at a given scale, however, depends in turn on the spectrum of excitations at larger scales [16].

Thus an accurate understanding of cosmological stretching of string modes is necessary to understand small-scale structure. Ref. [16] used a simple model which neglected interactions between different modes and assumed that once modes enter the horizon their physical amplitudes remain constant while their wavelengths are stretched. This model results in amplitude to wavelength ratios that decrease with decreasing wavelengths. The reason for this is that short modes have been in the horizon longer and have had more time to be stretched.

Here we have argued that there is a coupling between modes of very different wavelengths. While it is true that short modes are being stretched by the expansion of the Universe, the stretching of long modes that have recently entered the horizon produces a decrease in the arc length of the string which acts to decrease the wavelength of the short modes. The net result is that the wavelengths of short modes are not stretched as effectively by the expansion of the universe as the wavelength of long modes.

We have studied the evolution of the amplitudes numerically and found that, in contrast with the wavelength evolution, the amplitude evolution does not couple modes of different sizes: Regardless of the presence of other modes, a mode's physical amplitude remains fixed once it is well inside the horizon.

Although we have simulated only the case of smooth excitations in transverse directions, we believe our results are generic. The effect on the wavelength of the short mode can be understood analytically as explained in section II, and depends only on the loss of arc length in the long mode, and not on its shape or direction. We found that the amplitude of the short mode is insensitive to the presence of the long one, and we expect that principle to be generic. When there is a significant difference in wavelength, the long mode moves the string underlying the short mode as a whole, and we would not expect the direction of that motion to have any effect on the short mode evolution.

As compared to the previous simple model we expect the amplitude to wavelength ratio of the perturbations on cosmic strings to decrease more slowly as a function of decreasing frequency. The detailed consequences for the spectrum and the small-scale structure cutoff will be examined in detail elsewhere.

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